# 6. DETERMINANTS

## Let's study.

- Definition of Determinants
- Properties of Determinants
- Applications of Determinants
- Cramer's Rule
- Consistency of linear equations
- Area of triangle
- Collinearity of three points

## Let's recall.

In standard X, we have learnt to solve simultaneous equations in two variables using determinants of order two. We will now learn more about the determinants. It is useful in Engineering applications, Economics, etc.

> The concept of a determinant was discussed by the German Mathematician G.W. Leibnitz.
>
> Cramer developed the rule for solving linear equations using determinants.

The representation $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is defined as the determinant of order two. Numbers a, b, c, d are called elements of the determinant. In this arrangement, there are two rows and two columns.

$(a \quad b)$ is the $1^{st}$ Row $\quad$ $(c \quad d)$ is the $2^{nd}$ Row

$\begin{pmatrix} a \\ c \end{pmatrix}$ is the $1^{st}$ Column $\quad$ $\begin{pmatrix} b \\ d \end{pmatrix}$ is the $2^{nd}$ column

That is, four elements are enclosed between two vertical bars arranged in two rows and two columns.

The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is associated with the expression $ad - bc$,

$ad–bc$ is called the value of the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Note that $ad–bc$ is an algebraic expression in $a,b,c$ and $d$.

## Let's learn.

### 6.1 Determinant of order 3 :

**Definition** : The determinant of order 3 is a square arrangement of 9 elements enclosed between two vertical bars. The elements are arranged in 3 rows and 3 columns as given below.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$\begin{matrix} C_1 & C_2 & C_3 \end{matrix}$$

Here $a_{11}$, $a_{12}$, $a_{13}$ are element in row I ($R_1$)

$a_{21}$, $a_{22}$, $a_{23}$ are elements in row II ($R_2$)

$a_{31}$, $a_{32}$, $a_{33}$ are elements in row III ($R_3$)

Similarly $a_{11}$, $a_{21}$, $a_{31}$ are elements in column I ($C_1$) and so on.

Here $a_{ij}$ represents the element in $i^{th}$ row and $j^{th}$ column of the determinant.

For example : $a_{31}$ represents the element in $3^{rd}$ row and $1^{st}$ column.

In general, we denote determinant by D or A or Δ (delta) .

$$\text{For example, } D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Like a 2×2 determinant, 3×3 determinant is also an algebraic expression in terms of elements of the determinant. We find that expression by expanding the determinant.

### 6.1.1 Expansion of Determinant

There are six ways of expanding a determinant of order 3, corresponding to each of three rows ($R_1$, $R_2$ , $R_3$) and three columns ($C_1$, $C_2$, $C_3$). We give here the expansion by the 1st row of the determinant D

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The determinant can be expanded as follows:

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

**For example,**

$$D = \begin{vmatrix} 3 & -2 & 4 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

We expand the determinant as follows in two different ways.

$$(1)\ (D) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 3(-2-1) + 2(-1-0) + 4(1-0)$$

$$= 3(-3) + 2(-1) + 4(1)$$

$$= -9 -2 +4 = -11 \ \ +4 = -7$$

$$(2)\ D = (a_{11}.\, a_{22}\, .a_{33}) + (a_{12}.\, a_{23}\, .a_{31}) + (a_{13}.\, a_{32}\, .a_{21}) - (a_{13}.\, a_{22}\, .a_{31}) - (a_{12}.\, a_{21}\, .a_{33}) - (a_{11}.\, a_{32}\, .a_{23})$$

$$= (3\times2\times-1) + (-2\times1\times0) + (4\times1\times1) - (4\times2\times0) - (-2\times1\times-1) - (3\times1\times1)$$

$$= -6 + 0 + 4 - 0 - 2 - 3 = -11 + 4 = -7$$

**SOLVED EXAMPLES**

**Ex.1** Evaluate the following determinants :

1) $\begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix}$ 2) $\begin{vmatrix} -2 & 12 \\ 3 & 25 \end{vmatrix}$

3) $\begin{vmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{vmatrix}$ 4) $\begin{vmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{vmatrix}$

**Solution :**

1) $\begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} = 15 - 14 = 1$

2) $\begin{vmatrix} -2 & 12 \\ 3 & 25 \end{vmatrix} = -50 - 36 = -86$

3) $\begin{vmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{vmatrix}$

$$= 3\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 6\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 3(-1) - 2(-1) + 6(0) = -1$$

4) $\begin{vmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 3 & 3 \end{vmatrix}$

$$= 1(-8) - 3(-6) + 3(6)$$

$= -8+18+18=28$

**Ext. 2.** Find the value of $x$ if

1) $\begin{vmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 4 & 6 & 9 \end{vmatrix} = 0$ 2) $\begin{vmatrix} 1 & 1 & -x \\ x & -4 & 5 \\ x & -2 & 1 \end{vmatrix} = 0$

**Solution :**

1) $\begin{vmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 4 & 6 & 9 \end{vmatrix} = 0$

$\therefore$ $1(18 - 24) - x(9-16) + x^2 (6 - 8) = 0$

$\therefore$ $1(-6) - x(-7) + x^2 (-2) = 0$

$\therefore$ $-6 + 7x - 2x^2 = 0$

$\therefore$ $2x^2 - 7x + 6 = 0$

$\therefore$ $(2x - 3)(x - 2) = 0$

$\therefore$ $x = \dfrac{3}{2}$ or $x = 2$

2) $\begin{vmatrix} 1 & 1 & -x \\ x & -4 & 5 \\ x & -2 & 1 \end{vmatrix} = 0$

$\therefore$ $1(-4 + 10) - 1(x - 5x) - x(-2x + 4x) = 0$

$\therefore$ $1(6) - 1(-4x) - x(2x) = 0$

$\therefore$ $6 + 4x - 2x^2 = 0$

$\therefore$ $2x^2 - 4x - 6 = 0$

$\therefore$ $(2x + 2)(x - 3) = 0$

$\therefore$ $x = -1$ or $x = 3$

## EXERCISE 6.1

1) Evaluate the following determinants :

i) $\begin{vmatrix} 4 & 7 \\ -7 & 0 \end{vmatrix}$ ii) $\begin{vmatrix} 3 & -5 & 2 \\ 1 & 8 & 9 \\ 3 & 7 & 0 \end{vmatrix}$ iii) $\begin{vmatrix} 1 & i & 3 \\ i^3 & 2 & 5 \\ 3 & 2 & i^4 \end{vmatrix}$

iv) $\begin{vmatrix} 5 & 5 & 5 \\ 5 & 4 & 4 \\ 5 & 4 & 8 \end{vmatrix}$ v) $\begin{vmatrix} 2i & 3 \\ 4 & -i \end{vmatrix}$ vi) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

vii) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ viii) $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

2) Find the value of $x$ if

i) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

ii) $\begin{vmatrix} 2 & 1 & x+1 \\ -1 & 3 & -4 \\ 0 & -5 & 3 \end{vmatrix} = 0$ iii) $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$

3) Solve the following equations.

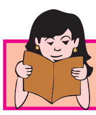
i) $\begin{vmatrix} x & 2 & 2 \\ 2 & x & 2 \\ 2 & 2 & x \end{vmatrix} = 0$ ii) $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

4) Find the value of $x$ if

$\begin{vmatrix} x & -1 & 2 \\ 2x & 1 & -3 \\ 3 & -4 & 5 \end{vmatrix} = 29$

5) Find $x$ and $y$ if $\begin{vmatrix} 4i & i^3 & 2i \\ 1 & 3i^2 & 4 \\ 5 & -3 & i \end{vmatrix} = x + iy$

where $i = \sqrt{-1}$



**Let's learn.**

## 6.2 PROPERTIES OF DETERMINANTS

In the previous section we have learnt how to expand a determinant. Now we will study some properties of determinants. They will help us to evaluate a determinant more easily.

**Property 1** - The value of determinant remains unchanged if its rows are turned into columns and columns are turned into rows.

Let D $= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1 .(b_2 c_3 - b_3 c_2) - b_1 (a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

$= a_1b_2 c_3 - a_1 b_3 c_2 - b_1 a_2 c_3 + b_1 a_3 c_2 + c_1 a_2 b_3 - c_1a_3b_2$ ———————— (i)

Let $D_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$

$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2c_1b_3 + a_3b_1c_2 - a_3c_1b_2$ ———————— (ii)

From (i) and (ii) D= $D_1$

For Example,

Let A $= \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 0 & 2 & 1 \end{vmatrix}$

$= 1\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix}$

$= 1(-1-4) - 2 (3-0) - 1(6-0)$

$= -5 - 6 - 6$

$= -17$ …………………………… (i)

By interchanging rows and columns of A we get the determinant $A_1$

$A_1 = \begin{vmatrix} 1 & 3 & 0 \\ 2 & -1 & 2 \\ -1 & 2 & 1 \end{vmatrix}$

$= 1\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} -3\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= 1(-1-4) - 3 (2+2) + 0 = -5 - 12 = -17$ ………………………………...(ii)

$\therefore$ A = $A_1$ from (i) and (ii)

**Property 2** - If any two rows (or columns) of a determinant are interchanged then the value of the determinant changes only in sign.

Let D $= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1 .(b_2 c_3 - b_3 c_2) - b_1 (a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

$= a_1b_2 c_3 - a_1 b_3 c_2 - b_1a_2c_3 + b_1a_3c_2 + c_1a_2b_3 - c_1a_3b_2$

Let $D_1$ = determinant obtained by interchanging first and second row of determinant D

$D_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_2 (b_1c_3 - b_3c_1) - b_2 (a_1c_3 - a_3c_1) + c_2( a_1b_3 - a_3b_1)$

$= a_2b_1c_3 - a_2b_3c_1 - b_2 a_1c_3 + b_2a_3c_1 + c_2 a_1b_3 - c_2 a_3b_1$

$= -[a_1b_2 c_3 - a_1 b_3 c_2 - b_1a_2c_3 + b_1a_3c_2 + c_1a_2b_3 - c_1a_3b_2]$

$= -$ D

For Example,

Let A $= \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 0 & 2 & 1 \end{vmatrix}$

$= 1(-1-4) - 2 (3- 0) - 1(6-0)$

$= -5-6-6$

$= -17$

Interchange 1st and 3rd rows. Then new determinant is given by $A_1$

$$A_1 = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$= 0(1-4) - 2(-3-2)+1(6+1)$

$= 0+ 10 + 7$

$= 17$

$\therefore A_1 = -A$

Interchange 2nd and 3rd column in A and let the new determinant obtained be $A_2$

$$\therefore A_2 = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

$= 1(4+1) +1(6)+2(3)$

$= 5+6+6 = 17 \quad \therefore A_2 = -A$

Note – We denote the interchange of rows by $R_i \leftrightarrow R_j$ and interchange of columns by $C_i \leftrightarrow C_j$

**Property 3** - If any two rows ( or columns) of a determinant are identical then the value of the determinant is zero

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Here first and second rows are identical.

$D = a_1 .(b_1 c_3 - b_3 c_1) - b_1 (a_1c_3 - a_3c_1 ) + c_1(a_1 b_3 - a_3b_1 )$

$= a_1b_1 c_3 - a_1b_3 c_1 - b_1a_1c_3 + b_1 a_3c_1 + c_1a_1 b_3 - c_1 a_3b_1$

$= 0$

$$\text{For Example, } A = \begin{vmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 0 & 1 & 2 \end{vmatrix}$$

$= 1(-2-2) -(-1) (2-0) + 2(1-0)$

$= -4 + 2 + 2 = -4 + 4 = 0$

Here in determinant A first and second rows are identical

**Property 4** - If each element of a row (or column) of a determinant is multiplied by a constant k then the value of the new determinant is k times the value of the original determinant.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= a_1 ( b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 ( a_2b_3 - a_3b_2)$ ................ (i)

$$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$$

$= a_1 (kb_2c_3 - kb_3c_2) - b_1 (ka_2c_3 - ka_3c_2) + c_1 (ka_2b_3 - ka_3b_2)$

$= k[a_1 (b_2c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2b_3 - a_3b_2)]$

$= k D$ (from i)

$$\text{For Example, } A = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$= 0 - 2(-3-2) + 1(6+1)$

$= + 10 + 7$

$= 17$

Multiply $R_2$ by 3 we get,

$$A_1 = \begin{vmatrix} 0 & 2 & 1 \\ 3\times3 & -1\times3 & 2\times3 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 & 1 \\ 9 & -3 & 6 \\ 1 & 2 & -1 \end{vmatrix} = 0 - 2(-9-6) + 1(18+3)$$

$= 30 + 21 = 51$

$= 3 \times 17$

$\therefore A_1 = 3 A$

**Remark** i) Using this property we can take out a common factor from any one row (or any one column) of the given determinant.

ii) If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio) then the value of the determinant is zero.

Example :

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 8 & 16 & 24 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 8\times1 & 8\times2 & 8\times3 \end{vmatrix} = 8\begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 1 & 2 & 3 \end{vmatrix}$$

(using property 4)

$= 8 \times 0$ (using property 3)

**Activity 6.1**

$$\text{Simplify : } A = \begin{vmatrix} 25 & 75 & 125 \\ 13 & 26 & 39 \\ 17 & 51 & 34 \end{vmatrix}$$

$$A = \square \times \square \times \square \begin{vmatrix} 1 & 3 & 5 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= \square\, [1 \times (\square - \square) - 1(6-15)$$

$$+1\,(\square - \square)]$$

$$= \square \times \square = \square$$

**Property 5 -** If each element of a row (or column) is expressed as the sum of two numbers then the determinant can be expressed as sum of two determinants

Example (i)

$$\begin{vmatrix} a_1+x_1 & b_1+y_1 & c_1+z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example (ii)

$$\begin{vmatrix} x_1+y_1+z_1 & p & l \\ x_2+y_2+z_2 & q & m \\ x_3+y_3+z_3 & r & n \end{vmatrix} = \begin{vmatrix} x_1 & p & l \\ x_2 & q & m \\ x_3 & r & n \end{vmatrix} + \begin{vmatrix} y_1 & p & l \\ y_2 & q & m \\ y_3 & r & n \end{vmatrix} + \begin{vmatrix} z_1 & p & l \\ z_2 & q & m \\ z_3 & r & n \end{vmatrix}$$

**Property 6 -** If a constant multiple of all elements of any row (or column) is added to the corresponding elements of any other row (or column ) then the value of new determinant so obtained is the same as that of the original determinant.

**Note -** In all the above properties we perform some operations on rows (or columns). We indicate these operations in symbolic form as below.

i) $R_i \leftrightarrow R_j$ means interchange $i^{th}$ and $J^{th}$ rows.

ii) $C_i \leftrightarrow C_j$ means interchange $i^{th}$ and $j^{th}$ columns.

iii) $R_i \rightarrow kR_i$ ( or $C_i \rightarrow kC_i$) means multiplying $i^{th}$ row (or $i^{th}$ column) by constant $k$.

iv $R_i \rightarrow R_i + kR_j$ ( or $C_i \rightarrow C_i + kC_j$) means change in $i^{th}$ row (or $i^{th}$ column) by adding $k$ multiples of corresponding elements of $j^{th}$ row (or $j^{th}$ column) in $i^{th}$ row (or $i^{th}$ column).

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Performing $R_1 \rightarrow R_1 + k.R_3$

$$A_1 = \begin{vmatrix} a_1+ka_3 & b_1+kb_3 & c_1+kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$A_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(by property 5)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(by property 4)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k(0)$$

($R_1$ and $R_3$ are identical)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = A$$

Example : Let $B = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix}$

$= 1(2–0) – 2(–1–0) + 3(–2–2)$

$= 2 + 2 – 12$

$= 4–12 = –8$ ————————(i)

Now, $B = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix}$

$R_1 \to R_1 + 2R_2$. Then

$$B_1 = \begin{vmatrix} 1+2(-1) & 2+2(2) & 3+2(0) \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$B_1 = \begin{vmatrix} -1 & 6 & 3 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix} = -1(2-0) - 6(-1-0) + 3(-2-2)$$

$= -2 + 6 - 12 = 6 - 14 = -8$ ————(ii)

From (i) and (ii) $B = B_1$

Remark – If more than one operations are to be done, make sure that the operations are done one at a time. Else there may be a mistake in calculations.

**The diagonal of a determinant :** The diagonal (main or principal diagonal) of a determinant A is collection of entries $a_{ij}$ where $i = j$

OR

The set of elements ( $a_{11}$ , $a_{22}$, $a_{33}$ , ....... $a_{nn}$) forms the diagonal of a determinant A

$$\text{e.g. } D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Here $a_{11}$, $a_{22}$, $a_{33}$ are element of diagonal.

**Property 7 -** (Triangle property) – If all the elements of a determinant above or below the diagonal are zero then the value of the determinant is equal to the product of its diagonal elements.

Verification:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3$$

For example, $A = \begin{vmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 2 \end{vmatrix} = a_1b_2c_3 = 3\times4\times2 = 24$

## SOLVED EXAMPLES

**Ext. 1.** Without expanding determinant find the value of :

i) $\begin{vmatrix} 3 & 4 & -11 \\ 2 & -1 & 0 \\ 5 & -2 & -1 \end{vmatrix}$ ii) $\begin{vmatrix} 17 & 18 & 19 \\ 20 & 21 & 22 \\ 23 & 24 & 25 \end{vmatrix}$

**Solution :**

i) Let $D = \begin{vmatrix} 3 & 4 & -11 \\ 2 & -1 & 0 \\ 5 & -2 & -1 \end{vmatrix}$

Performing $R_1 \to R_1 + 4R_2$

$$= \begin{vmatrix} 11 & 0 & -11 \\ 2 & -1 & 0 \\ 5 & -2 & -1 \end{vmatrix}$$

Taking 11 common from $R_1$.

$$= 11\begin{vmatrix}1 & 0 & -1\\ 2 & -1 & 0\\ 5 & -2 & -1\end{vmatrix} \quad R_3 \to R_3 - 2R_2$$

$$= 11\begin{vmatrix}1 & 0 & -1\\ 2 & -1 & 0\\ 1 & 0 & -1\end{vmatrix}$$

$= 11\,(0) = 0....$

$\because$ $R_1$ and $R_3$ are indentical.

ii) Let D $= \begin{vmatrix}17 & 18 & 19\\ 20 & 21 & 22\\ 23 & 24 & 25\end{vmatrix}$

Performing $C_2 \to C_2 - C_1$

$$= \begin{vmatrix}17 & 1 & 19\\ 20 & 1 & 22\\ 23 & 1 & 25\end{vmatrix}$$

Performing $C_3 \to C_3 - 2C_2$

$$= \begin{vmatrix}17 & 1 & 17\\ 20 & 1 & 20\\ 23 & 1 & 23\end{vmatrix}$$

$= 0$, since $C_1$ and $C_3$ are identical.

**Ex 2.** By using properties of determinant show that

i) $\begin{vmatrix}p & q & r\\ a & b & c\\ x & y & z\end{vmatrix} = \begin{vmatrix}r & p & q\\ c & a & b\\ z & x & y\end{vmatrix}$

ii) $\begin{vmatrix}x+y & y+z & z+x\\ z+x & x+y & y+z\\ y+z & z+x & x+y\end{vmatrix} = 2\begin{vmatrix}x & y & z\\ z & x & y\\ y & z & x\end{vmatrix}$

**Solution :**

i) LHS $= \begin{vmatrix}p & q & r\\ a & b & c\\ x & y & z\end{vmatrix}$ Performing $C_1 \leftrightarrow C_3$

$$= (-1)\begin{vmatrix}r & p & q\\ c & b & a\\ z & y & x\end{vmatrix} \quad \text{Performing } C_2 \leftrightarrow C_3$$

$$= (-1)(-1)\begin{vmatrix}r & p & q\\ c & a & b\\ z & x & y\end{vmatrix}$$

$$= \begin{vmatrix}r & p & q\\ c & a & b\\ z & x & y\end{vmatrix} = \text{RHS}$$

ii) RHS $= \begin{vmatrix}x+y & y+z & z+x\\ z+x & x+y & y+z\\ y+z & z+x & x+y\end{vmatrix}$ $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix}2(x+y+z) & 2(x+y+z) & 2(x+y+z)\\ z+x & x+y & y+z\\ y+z & z+x & x+y\end{vmatrix}$$

$R_1 \to R_1 - R_3$

$$= 2\begin{vmatrix}x & y & z\\ z+x & x+y & y+z\\ y+z & z+x & x+y\end{vmatrix} \quad R_2 \to R_2 - R_1$$

$$= 2\begin{vmatrix}x & y & z\\ z & x & y\\ y+z & z+x & x+y\end{vmatrix} \quad R_3 \to R_3 - R_2$$

$$= 2\begin{vmatrix}x & y & z\\ z & x & y\\ y & z & x\end{vmatrix} = \text{RHS}$$

**Ex 3.** Solve the equation $\begin{vmatrix} x+1 & x+2 & 3 \\ 3 & x+2 & x+1 \\ x+1 & 2 & x+3 \end{vmatrix} = 0$

**Solution :**

$$\begin{vmatrix} x+1 & x+2 & 3 \\ 3 & x+2 & x+1 \\ x+1 & 2 & x+3 \end{vmatrix} = 0 \quad R_2 \to R_2 - R_1$$

$$\therefore \begin{vmatrix} x+1 & x+2 & 3 \\ 2-x & 0 & x-2 \\ x+1 & 2 & x+3 \end{vmatrix} = 0 \quad R_3 \to R_3 - R_1$$

$$\therefore \begin{vmatrix} x+1 & x+2 & 3 \\ 2-x & 0 & x-2 \\ 0 & -x & x \end{vmatrix} = 0 \quad C_2 \to C_2 + C_3$$

$$\therefore \begin{vmatrix} x+1 & x+5 & 3 \\ 2-x & x-2 & x-2 \\ 0 & 0 & x \end{vmatrix} = 0$$

$$\therefore x(x-2)\begin{vmatrix} x+1 & x+5 & 3 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$\therefore x(x - 2)[(x + 1)(1) - (x + 5)(-1) + 3(0)] = 0$

$\therefore x(x - 2)(2x + 6) = 0$

$\therefore 2x(x - 2)(x + 3) = 0$

$\therefore x = 0$ or $x = 2$ or $x = -3$

## EXERCISE 6.2

1) Without expanding evaluate the following determinants.

i) $\begin{vmatrix} 1 & a & a+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ ii) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ iii) $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

2) Using properties of determinant show that

$$\begin{vmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{vmatrix} = 4abc$$

3) Solve the following equation.

$$\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$$

4) If $\begin{vmatrix} 4+x & 4-x & 4-x \\ 4-x & 4+x & 4-x \\ 4-x & 4-x & 4+x \end{vmatrix} = 0$ then find the values of $x$

5) Without expanding determinants show that

$$\begin{vmatrix} 1 & 3 & 6 \\ 6 & 1 & 4 \\ 3 & 7 & 12 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 & 3 \\ 2 & 1 & 2 \\ 1 & 7 & 6 \end{vmatrix} = 10\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 7 \\ 3 & 2 & 6 \end{vmatrix}$$

6) Without expanding determinants find the value of

i) $\begin{vmatrix} 10 & 57 & 107 \\ 12 & 64 & 124 \\ 15 & 78 & 153 \end{vmatrix}$ ii) $\begin{vmatrix} 2014 & 2017 & 1 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 1 \end{vmatrix}$

7) Without expanding determinant prove that

i) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$

ii) $\begin{vmatrix} 1 & yz & y+z \\ 1 & zx & z+x \\ 1 & xy & x+y \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

## 6.4 APPLICATIONS OF DETERMINANTS

### 6.4.1 Cramer's Rule

In linear algebra Cramer's rule is an explicit formula for the solution of a system of linear equations in many variables. In previous class we have studied this with two variables. Our goal here is to extend the application of Cramer's rule to three equations in three variables (unknowns) . Variables are usually denoted by $x$, $y$ and $z$.

Theorem – If
$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

are three linear equations in three variables $x$, $y$ and $z$; $a_i$ , $b_i$ , $c_i$ are coefficients of $x,y,z$ and $d_i$ are constants. Then solutions of the equations are

$$x = \frac{Dx}{D}, y = \frac{Dy}{D}, z = \frac{Dz}{D}$$

provided D $\neq$ 0 and where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad Dy = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$Dx = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad Dz = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Remark : If D = 0 then there is no unique solution for the given system of equations.

## SOLVED EXAMPLES

**1)** Solve the following equations using Cramer's Rule

1) $2x – y + z = 1, \quad x + 2y + 3z = 8, \quad 3x + y – 4z = 1$

2) $x + y – z = 2, \quad x – 2y + z = 3, \quad 2x – y – 3z = –1$

**Solution :**

1) Given system is

$2x – y + z = 1$

$x + 2y + 3z = 8$

$3x + y – 4z = 1$

$$\therefore D = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{vmatrix}$$

$$= 2(–8 – 3) + 1(–4 – 9) + 1(1 – 6)$$
$$= 2(– 11) + 1(–13) + 1(–5) = –40$$

$$\therefore D_x = \begin{vmatrix} 1 & -1 & 1 \\ 8 & 2 & 3 \\ 1 & 1 & -4 \end{vmatrix}$$

$$= 1(–8 – 3) + 1(–32 – 3) + 1(8 – 2)$$
$$= –11 – 35 + 6 = –40$$

$$\therefore D_y = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 3 & 1 & -4 \end{vmatrix}$$

$$= 2(–32 –3) –1(–4 – 9) + 1(1 – 24)$$
$$= –70 + 13 – 23 = –80$$

$$\therefore D_z = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 8 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= 2(2 – 8) + 1(1 – 24) + 1(1 – 6)$$
$$= –12 – 23 – 5 = –40$$

$$x = \frac{D_x}{D} = \frac{-40}{-40} = 1 \quad y = \frac{D_y}{D} = \frac{-80}{-40} = 2$$

$$z = \frac{D_z}{D} = \frac{-40}{-40} = 1, \quad \therefore x = 1, \; y = 2, \; z = 1$$

2) Given system is

$x + y – z = 2; \quad x – 2y + z = 3;$

$2x – y – 3z = –1$

$$\therefore D = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= 1(6 + 1) – 1(–3 – 2)–1(–1 + 4)$$
$$= 7 + 5 – 3 = 9$$

$$D_x = \begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \\ -1 & -1 & -3 \end{vmatrix}$$

$$= 2(7) – 1(–8)–1(– 5) = 14 + 8 + 5 = 27$$

$$D_y = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= 1(-8) - 2(-5) - 1(-7) = -8 + 10 + 7 = 9$$

$$\therefore D_z = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= 1(5) - 1(-7) + 2(3) = 5 + 7 + 6 = 18$$

$$x = \frac{D_x}{D} = \frac{27}{9} = 3 \quad y = \frac{D_y}{D} = \frac{9}{9} = 1$$

$$z = \frac{D_z}{D} = \frac{18}{9} = 2$$

$\therefore \quad x = 3, y = 1, z = 2$

### 6.4.2 Consistency of three linear equations in two variables

Consider the system of three linear equations in two variables x and y

$$\left.\begin{aligned} a_1x + b_1y + c_1 &= 0 \\ a_2x + b_2y + c_2 &= 0 \\ a_3x + b_3y + c_3 &= 0 \end{aligned}\right\} \ldots . \text{(I)}$$

These three equations are said to be consistent if they have a common solution.

The ***necessary*** condition for system (I) to be consistent is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

**SOLVED EXAMPLES**

**Ex. 1)** Show that the equations are consistent $3x + 4y = 11$, $2x - y = 0$ and $5x - 2y = 1$

**Solution :** By condition of consistency consider

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 3 & 4 & -11 \\ 2 & -1 & 0 \\ 5 & -2 & -1 \end{vmatrix}$$

$$= 3(1 - 0) - 4(-2 - 0) - 11(-4 + 5) = 3 + 8 - 11$$
$$= 11 - 11 = 0$$

$\therefore$ the given system of equations is consistent

**Ex. 2)** Show that the following equations are not consistent

$x+2y = 1$, $x-y = 2$ and $x-2y = 0$

**Solution :** For consistency of equations, consider

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -2 & 0 \end{vmatrix}$$

$$= 1(0+4) - 2(0-2) - 1(-2+1) = 4+4+1 = 9 \neq 0$$

$\therefore$ the given system of equations is not consistent.

**Note :** Consider the following equations

$$2x + 2y = -3, \quad x + y = -1, \quad 3x + 3y = -5$$

The given equations in the standard form are

$$2x + 2y + 3 = 0, \quad x + y + 1 = 0, \quad 3x + 3y + 5 = 0$$

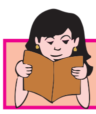
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix}$$

$= 0 \qquad (\text{As } R_1 = R_2)$

The slopes of the lines $2x + 2y + 2 = 0$ and $x + y + 1 = 0$ are equal and each is $(-1)$

So, the slope of each of these lines is - 1. Threrefore, the lines are parallel. So they do not have a common solution and hence the system of equations is not consistent.

Thus D = 0 is not a sufficient condition for the consistency of three equations in two variables.



**Let's learn.**

### 6.4.3 Area of a triangle and Collinearity of three points.

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of triangle ABC then the area of the triangle is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

**Remark:**

i) Area is a positive quantity. Hence we always take the absolute value of the determinant.

ii) If area is given, consider both positive and negative values of determinant for calculation of unknown co–ordinates.

iii) If the value of the determinant is zero then the area of Δ ABC is zero. It implies that the points A, B and C are **collinear**.

## SOLVED EXAMPLES

**Q.1)** Find the area of the triangle whose vertices are $P(2, -6)$ , $Q(5, 4)$, $R(-2, 4)$

**Solution:** Let, $P(2,-6) \equiv (x_1,y_1)$, $Q(5,4) \equiv (x_2,y_2)$, $R(-2,4) \equiv (x_3,y_3)$.

Area of the triangle $PQR$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\ [2(0) + 6(7) + 1(28)]$$

$$= \frac{1}{2}\ [70]=35 \quad \text{square unit.}$$

**Q. 2)** Using determinant show that following points are collinear.

$A(2,3)$, $B(-1,-2)$, $C(5,8)$

**Solution:**

$$A(\Delta ABC) = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & -2 & 1 \\ 5 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\ [2(-10) - 3(-6) + 1(2)]$$

$$= \frac{1}{2}\ [-20 + 18 + 2\ ] = 0$$

∴ Points $A$, $B$, $C$ are collinear.

**Q.3)** If $P(3,-5)$, $Q(-2,k)$, $R(1,4)$ are vertices of triangle $PQR$ and its area is $\frac{17}{2}$ square unit then find the value of $k$.

**Solution :** Let $P(3,-5) \equiv (x_1,y_1)$, $Q(-2,k) \equiv (x_2,y_2)$, $R(1,4) \equiv (x_3,y_3)$.

$$A(\Delta PQR) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 3 & -5 & 1 \\ -2 & k & 1 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\ [3(k-4) + 5(-3) + 1(-8-k)]$$

$$= \frac{1}{2}\ [3k - 12 - 15 - 8 - k] = \frac{1}{2}\ [2k - 35]$$

But area of triangle $PQR$ is $\frac{17}{2}$

$$\therefore \frac{1}{2}\ [2k - 35] = \pm\frac{17}{2}$$

$\therefore\ 2k - 35 = \pm 17$

$\therefore\ 2k = 52$ or $2k = 18$

$\therefore\ k = 26$ or $k = 9$

**Q.4)** Find the area of the triangle whose vertices are A(–2, –3) , B(3, 2) and C(–1, –8)

**solution :** Given $(x_1, y_1) \equiv (-2, -3)$, $(x_2, y_2) \equiv (3, 2)$, and $(x_3, y_3) \equiv (-1\ , -8)$

We know that area of a triangle

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\ [-2(2 + 8) + 3(3 + 1) + 1(-24 + 2)]$$

$$= \frac{1}{2}\ [-20 + 12 - 22] = \frac{1}{2}\ [-42 + 12]$$

$= \frac{1}{2}[-30] \quad = -15$

But Area is positive.

$\therefore$ Area of triangle = 15 square unit

**Q.5)** If the area of a triangle with vertices $P(-3, 0)$, $Q(3, 0)$ and $R(0, k)$ is 9 square unit then find the value of k.

**Solution :** Given $(x_1, y_1) \equiv (-3, 0)$, $(x_2, y_2) \equiv (3, 0)$ and $(x_3, y_3) \equiv (0, k)$ and the area of the triangle is 9 sq. unit.

We know that area of a triangle

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\therefore \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix} = \pm 9$$

(Area is positive but the value of the determinant can be positive or negative)

$$\therefore \frac{1}{2}[-3(0-k) + 1(3k-0)] = \pm 9$$

$$\therefore \frac{1}{2}[6k] = \pm 9,$$

$$\therefore 6k = \pm 18,$$

$$\therefore k = \pm 3$$

## EXERCISE 6.3

1) Solve the following equations using Cramer's Rule.

i) $x + 2y - z = 5$, $2x - y + z = 1$, $3x + 3y = 8$

ii) $2x - y + 6z = 10$, $3x + 4y - 5z = 11$, $8x - 7y - 9z = 12$

iii) $11x - y - z = 31$, $x - 6y + 2z = -26$, $x + 2y - 7z = -24$

iv) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -2$

$\frac{1}{x} - \frac{2}{y} + \frac{1}{z} = 3$

$\frac{2}{x} - \frac{1}{y} + \frac{3}{z} = -1$

v) $\frac{2}{x} - \frac{1}{y} + \frac{3}{z} = 4$, $\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 2$, $\frac{3}{x} + \frac{1}{y} - \frac{1}{z} = 2$

2) An amount of Rs. 5000 is invested in three plans at rates 6%, 7% and 8% per annuam respectively. The total annual income from these investments is Rs 350. If the total annual income from first two investments is Rs. 70 more than the income from the third, find the amount invested in each plan by using Cramer's Rule.

3) Show that following equations are consistent.

$2x + 3y + 4 = 0$, $x + 2y + 3 = 0$,
$3x + 4y + 5 = 0$

4) Find $k$ if the following equations are consistent.

i) $x + 3y + 2 = 0$, $2x + 4y - k = 0$, $x - 2y - 3k = 0$

ii) $(k-1)x + (k-1)y = 17$, $(k-1)x + (k-2)y = 18$, $x + y = 5$

5) Find the area of the triangle whose vertices are :

i) (4,5), (0,7), (–1,1)

ii) (3,2,), (–1,5), (–2,–3)

iii) (0,5), (0,–5), (5,0)

6) Find the value of k if the area of the triangle with vertices $A(k,3)$, $B(-5,7)$, $C(-1,4)$ is 4 square unit.

7) Find the area of the quadrilateral whose vertices are $A(-3,1)$, $B(-2,-2)$, $C(4,1)$, $D(2,3)$.

8) By using determinant show that following points are collinear.

$P(5,0)$, $Q(10,-3)$, $R(-5,6)$

9) The sum of three numbers is 15. If the second number is subtracted from the sum of first and third numbers then we get 5. When the third number is subtracted from the sum of twice the first number and the second number, we get 4. Find the three numbers.

## Let's remember!

1) The value of determinant of order 3×3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

2) **Properties of determinant**

(i) The value of determinant remains unchanged if its rows are turned into columns and columns are turned into rows.

(ii) If any two rows (or columns) of a determinant are interchanged then the value of the determinant changes its sign.

(iii) If any two rows (or columns) of a determinant are identical then the value of the determinant is zero

(iv) If each element of a row (or column) of a determinant is multiplied by a constant k then the value of the new determinant is k times the value of the old determinant

(v) If each element of a row (or column) is expressed as the sum of two numbers then the determinant can be expressed as a sum of two determinants

(vi) If a constant multiple of all elements of a row (or column) is added to the corresponding elements of any other row (or column) then the value of new determinant so obtained is the same as that of the original determinant.

(vii) (Triangle property) – If all element of the determinant above or below the main diagonal are zero then the value of the determinant is equal to product of its diagonal elements.

3) Solve the system of linear equation using Cramer's Rule.

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D} \text{ provided } D \neq 0$$

4) Consistency of three equations.

If $a_1x + b_1y + c_1 = 0$ $a_2x + b_2y + c_2 = 0$

$a_3x + b_3y + c_3 = 0$ are consistent, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

5) Area of a triangle, the vertices of which are $(x_1, y_1), (x_2,y_2)\ (x_3,y_3)$; is

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

6) Points $(x_1, y_1), (x_2,y_2)\ (x_3,y_3)$ are collinear if and only if

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

## MISCELLANEOUS EXERCISE - 6

**Q.1)** Evaluate i) $\begin{vmatrix} 2 & -5 & 7 \\ 5 & 2 & 1 \\ 9 & 0 & 2 \end{vmatrix}$ ii) $\begin{vmatrix} 1 & -3 & 12 \\ 0 & 2 & -4 \\ 9 & 7 & 2 \end{vmatrix}$

**Q.2)** Find the value of $x$ if

i) $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & -5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ ii) $\begin{vmatrix} 1 & 2x & 4x \\ 1 & 4 & 16 \\ 1 & 1 & 1 \end{vmatrix} = 0$

**Q.3)** By using properties of determinant prove that $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

**Q.4)** Without expanding determinant as far as possible, show that

i) $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0$

ii) $\begin{vmatrix} xa & yb & zc \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ a & b & c \\ bc & ca & ab \end{vmatrix}$

iii) $\begin{vmatrix} l & m & n \\ e & d & f \\ u & v & w \end{vmatrix} = \begin{vmatrix} n & f & w \\ l & e & u \\ m & d & v \end{vmatrix}$

iv) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$

**Q.5)** Solve the following linear equations by Cramer's Rule.

i) $2x-y+z = 1$, $x+2y+3z = 8$, $3x+y-4z = 1$

ii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -2$

$\frac{1}{x} - \frac{2}{y} + \frac{1}{z} = 3$

$\frac{2}{x} - \frac{1}{y} + \frac{3}{z} = -1$

iii) $x-y+2z=7$ , $3x+4y-5z=5$ , $2x-y+3z=12$

**Q.6)** Find the value of k if the following equation are consistent.

i) $3x+y-2=0$, $kx+2y-3=0$ and $2x-y=3$

ii) $kx + 3y + 4 = 0$

$x + ky + 3 = 0$

$3x + 4y + 5 = 0$

**Q.7)** Find the area of triangle whose vertices are

i) A( –1,2) ,B(2,4) ,C(0,0)

ii) P(3,6) , Q(–1,3) , R (2,–1)

iii) L(1,1) , M (–2,2), N (5,4)

**Q.8)** Find the value of *k*.

i) if area of triangle Δ PQR is 4 square unit and vertices are $P(k, 0)$, $Q(4, 0)$, $R(0, 2)$

ii) if area of Δ LMN is 33/2 square unit and vertices are $L$ (3,–5), $M(-2,k)$, $N$ (1,4)

## ACTIVITIES

**Activity 6.1 :**

Apply cramer's Rule for the following linear equation $x + y - z = 1$, $8x + 3y - 6z = 1$, $-4x - y + 3z = 1$.

**Solution:**

$D = \begin{vmatrix} 1 & 1 & -1 \\ 8 & 3 & -6 \\ -4 & -1 & 3 \end{vmatrix} = \square$

$Dx = \begin{vmatrix} \square & 1 & -1 \\ \square & 3 & -6 \\ \square & -1 & 3 \end{vmatrix} = \square$

$Dy = \begin{vmatrix} 1 & \square & -1 \\ 8 & \square & -6 \\ -4 & \square & 3 \end{vmatrix} = \square$

$Dz = \begin{vmatrix} 1 & 1 & \square \\ 8 & 3 & \square \\ -4 & -1 & \square \end{vmatrix} = \square$

$x = \frac{Dx}{D} = \frac{\square}{\square} = \square$

$y = \frac{\square}{D} = \frac{\square}{\square} = \square$

$z = \frac{Dz}{D} = \frac{\square}{\square} = \square$

### Activity 6.2 :

**Fill in the blanks in the steps of solution of the following problem and complete it.**

The cost of 4 kg potato, 3 kg wheat and 2 kg rice is Rs. 150. The cost of 1 kg potato, 2 kg wheat and 3 kg rice is Rs. 125. The cost of 6 kg potato, 2 kg wheat and 3 kg rice is Rs. 175. Find the cost of each item per kg by using Cramer's Rule.

**Solution :** Let x,y,z be the costs of potato, wheat and rice per kg respectively. The given information can be written in equation form as

$4x + 3y + 2z = \ldots$
$x + \ldots y + 3z = 125$
$\ldots x + 2y + 3z = 175$

$$D = \begin{vmatrix} 4 & 3 & 2 \\ 1 & \ldots & 3 \\ 6 & 2 & 3 \end{vmatrix} = 25$$

$$D_x = \begin{vmatrix} \ldots & 3 & 2 \\ 125 & 2 & 3 \\ 175 & 2 & 3 \end{vmatrix} = 250$$

$$D_y = \begin{vmatrix} 4 & 150 & 2 \\ 1 & 125 & 3 \\ 6 & 175 & 3 \end{vmatrix} = \ldots$$

$$D_z = \begin{vmatrix} 4 & 3 & 150 \\ 1 & \ldots & 125 \\ 6 & 2 & 175 \end{vmatrix} = 625$$

$$x = \frac{D_x}{D} = \ldots \quad y = \frac{D_y}{D} = \ldots \quad z = \frac{D_z}{D} = \ldots$$

### Activity 6.3 :

Find the equation of the line joining the points $P(2,-3)$ & $Q(-4,1)$ using determinants.

**Solution:** $P(2,-3)$ & $Q(-4,1)$, $R(\square, \square)$

$\because$ Area of $\Delta$ PQR = 0

$$\square \begin{vmatrix} \square & \square & 1 \\ \square & \square & 1 \\ \square & \square & 1 \end{vmatrix} = 0$$

$\square = 0$ is the eq$^n$ of line

### Activity 6.4 :

Find k, if the following equations are consistent.

$x + y = 0,\ kx - 4y + 5 = 0,\ kx - 2y + 1 = 0$

**Solution:**

$$\begin{vmatrix} \square & \square & \square \\ \square & \square & 5 \\ \square & \square & 1 \end{vmatrix} = 0$$

$6 + \square = 0$

$4k = \square$

$k = -\square$

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